Math 1545 Calculus II — Prof. Paul Bailey

Problem Set 1 Solutions — February 20, 2006

Problem 1. Let $\gamma : [-\pi/2, \pi/2] \to \mathbb{R}^2$ be given by $\gamma(t) = (x(t), y(t))$, where $x(t) = \sec t$ and $y(t) = \tan t$. Let D be the region in $\mathbb{R}^2$ bounded by the image of γ and the lines $y = 0$, $y = 1$, and $y = x$.

(a) Find an algebraic expression which relates $x(t)$ and $y(t)$.

(b) Sketch the graph of the image of γ.

Let D be the region in $\mathbb{R}^2$ bounded by the image of γ and the lines $y = 0$, $y = 1$, and $y = x$.

(c) Find the volume of the solid obtained by revolving D around the y-axis.

(d) Find the volume of the solid obtained by revolving D around the x-axis.

Solution. The trigonometric identity $1 + \tan^2 t = \sec^2 t$ is well known. Since $x(t) = \sec t$ and $y(t) = \tan t$, we have $1 + y^2 = x^2$, which we may rewrite as

$$x^2 - y^2 = 1.$$

This is the equation of a hyperbola. The branch of the hyperbola which is parameterized by γ is the right branch; it passes through the point $(1, 0)$ and is asymptotic to the lines $y = \pm x$.

The right branch of our hyperbola is most conveniently viewed as the graph of x as a function of y; that is, consider the parabola to be the graph of the function $x = \sqrt{1 + y^2}$.

To find the volume V_y of D revolved around the y-axis, we use washers. Our variable of integration is y. The outer radius of a washer is $r_2 = \sqrt{1 + y^2}$ and the inner radius is $r_1 = y$. The region D is bounded above by $y = 0$ and above by $y = 1$, so these are the limits of integration. We have

$$\begin{aligned} V_y &= \int_0^1 \pi(r_2^2 - r_1^2)\, dy \\ &= \int_0^1 \pi(1 + y^2 - y^2)\, dy \\ &= \pi x \Big|_0^1 \\ &= \pi. \end{aligned}$$

To find the volume V_x of D revolved around the x-axis, we use shells. This allows us to keep y as the variable of integration and 0 to 1 as the limits. The height of a shell is $h = \sqrt{1 + y^2} - y$, and the radius of a shell is $r = y$, so we have

$$\begin{aligned} V_x &= \int_0^1 2\pi r h\, dy \\ &= \int_0^1 2\pi y(\sqrt{1 + y^2} - y)\, dy \\ &= 2\pi \Big[\int_0^1 y\sqrt{1 + y^2} - y^2\, dy \Big] \\ &= 2\pi \Big[\frac{1}{2}(\frac{2}{3})(1 + y^2)^{3/2} - \frac{1}{3}y^3 \Big]_0^1 \\ &= \frac{2\pi}{3}[2\sqrt{2} - 1 - 1] \\ &= \frac{4\pi}{3}(\sqrt{2} - 1). \end{aligned}$$

□

Definition 1. The *hyperbolic sine* and *hyperbolic cosine* functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Problem 2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be given by $\gamma(t) = (x(t), y(t))$, where $x(t) = \cosh t$ and $y(t) = \sinh t$.

(a) Find an algebraic expression which relates $x(t)$ and $y(t)$.

(b) Sketch the graph of the image of γ.

Let D be the region in $\mathbb{R}^2$ bounded by the image of γ and the lines $y = 0$, $y = 1$, and $y = x$.

(c) Find the volume of the solid obtained by revolving D around the y-axis.

(d) Find the volume of the solid obtained by revolving D around the x-axis.

Solution. In this case we use the identity $\cosh^2 t - \sinh^2 t = 1$. Since $x = \cosh t$ and $y = \sinh t$, the equation of the image of γ is $x^2 - y^2 = 1$, which is identical to the equation from the first problem. Thus, the images of these paths are identical, even though they are parameterized at different rates. Since the volumes asked for depend only on the curve and not on the parametrization, they are the same in this case as they are in Problem 1. □

Problem 3. Let $f : [0, 1] \to \mathbb{R}$ by given by $f(x) = \cosh(x)$. Find the area of the surface generated by revolving the graph of f around the x-axis.

Solution. The surface area A is given by

$$\int_a^b 2\pi r \, ds.$$

Since we are revolving around the x-axis, the $r = y = \cosh x$ and $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$. Now $f'(x) = \sinh x$, so $1 + (f'(x))^2 = 1 + \sinh^2 x = \cosh^2 x$, whence $ds = \cosh x \, dx$. The limits of integration are given in the problem as $a = 0$ and $b = 1$. Thus

$$\begin{aligned} A &= \int_0^1 2\pi \cosh^2 x \, dx \\ &= 2\pi \int_0^1 \frac{e^{2x} + 2 + e^{-2x}}{4} \, dx \\ &= \frac{\pi}{2} \left[\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} \right]_0^1 \\ &= \frac{\pi}{4} [e^2 + 4 - e^{-2} - 1 - 0 + 1] \\ &= \frac{\pi(e^2 + 4 - e^{-2})}{4}. \end{aligned}$$

□

Problem 4. Let $f(x) = \log_x e$.

(a) Find the domain and image of $f(x)$.

(b) Find $f'(x)$.

(c) Sketch the graph of $f(x)$.

Solution. The method here is to rewrite $f(x)$ in a more understandable form. We use the logarithmic change of base formula,

$$\log_b u = \frac{\log_a u}{\log_a b}.$$

Letting $a = e$, $b = x$, and $u = e$, this gives

$$f(x) = \frac{\log e}{\log x} = \frac{1}{\log x}.$$

So, perhaps surprisingly, this turns out to be the reciprocal of the natural logarithm. Thus is domain consists of the values of x in the domain of logarithm for which the log is not zero, and the image is all nonzero real numbers. The derivative is

$$f'(x) = \frac{d}{dx} \log^{-1} x = -\frac{1}{x \log^2 x}.$$

□